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A crossed-fields arrangement for use in an electron–positron *g*-factor comparison experiment

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Abstract. The motion of relativistic electrons close to the xy plane when subjected to static magnetic and electric fields $(0, 0, B_0)$ and $(0, E_0(1-ay), E_0az)$ is analysed, B_0, E_0 and a being positive constants. The focusing properties of the fields are investigated, and an expression is derived for the drift velocity of the particles in the direction of the positive x axis. The dependence of the latter quantity on the initial direction of motion and on the energy of the particles is discussed. Finally, the practical realization of the required electric field is considered. Throughout, the relevance of the analysis to the authors' electron–positron g-factor comparison experiment is emphasized.

1. Introduction

The authors are at present concerned with an experiment, the aim of which is to compare the g-factor anomalies of the positron and the electron, using the method of spin precession in a uniform magnetic field. Polarized particles (electrons or positrons) of known initial helicity execute cyclotron orbital motion for a controlled time in a plane perpendicular to a static magnetic field. The helicity of the particles is then measured by some suitable analyser. A determination of final helicity as a function of time spent in the magnetic field allows the g-factor anomaly to be determined directly (Wilkinson and Crane 1963, Rich and Crane 1966).

In the present experiment polarized particles emitted from a beta-active source are used. The time spent between source and analyser is controlled by the application of a weak electric field which, in the plane of the motion, is perpendicular to a uniform magnetic field. This produces the well-known drifting of the almost circular orbits in a direction orthogonal to both fields, the drift velocity (and thus the time spent by the particles in the magnetic field) being determined by the magnitude of the electric field.

The method is, therefore, very similar to the electron g-factor experiment of Farago *et al.* (1963). However, focusing, which is necessary to trap the particles in the magnetic field, is produced by distorting slightly the otherwise uniform electric field: in Farago's experiment, focusing was achieved by shaping the magnetic field.

In this paper we investigate the motion of an electron in an arrangement of fields which is suitable for a g-factor comparison experiment. The focusing properties of the system are considered, and an expression for the drift velocity of the electron is derived. The latter quantity, and especially its dependence on the direction of emission of the particle from the source (spherical aberration) and on particle energy (chromatic aberration), is of fundamental importance in the present experiment. The helicity of the beam of particles is analysed at a fixed distance from the source. Since the measured helicity varies linearly with the time spent by the particles between source and analyser, the above-mentioned aberrations produce depolarization of the analysed beam. At best, such depolarization implies an increase in running time to achieve a specified precision; at worst, it renders the experiment impossible.

2. Analysis of particle motion

The electric and magnetic fields are specified by

$$\mathbf{E} = (0, E_0(1-ay), E_0az); \qquad \mathbf{B} = (0, 0, B_0)$$

where E_0, B_0 and a are positive constants, and $a|y| < 1$ (see § 6). For a non-relativistic particle such an electric field will produce perfect focusing in the z direction, without affecting the xy motion in any way. We wish to analyse the motion close to the xy plane of a relativistic particle emitted from the source at an angle ϕ to the negative y direction, the source being at the origin of coordinates. This is shown in figure 1.

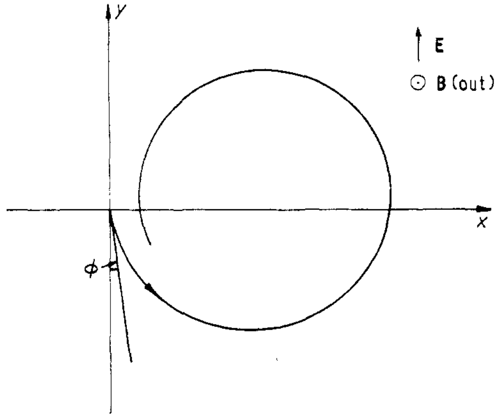

Figure 1. Orientation of axes and field directions, and showing a typical orbit.

The equation of motion of an electron of charge $-e$ and mass m in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\frac{d}{dt}(\gamma m\mathbf{v}) = -e(\mathbf{E}+\mathbf{v}\wedge\mathbf{B}) \tag{1}$$

where $\mathbf{v}$ is the electron velocity, $\gamma = (1-v^2/c^2)^{-1/2}$, c is the velocity of light, and all quantities are measured in the laboratory frame.

Since γ is not constant, equation (1) may be rewritten

$$m(\gamma\dot{\mathbf{v}}+\dot{\gamma}\mathbf{v}) = -e(\mathbf{E}+\mathbf{v}\wedge\mathbf{B}). \tag{2}$$

Multiplying equation (2) scalarly by $\mathbf{v}$, and using the identity

$$\dot{\gamma} \equiv \frac{\gamma^3\mathbf{v}\cdot\dot{\mathbf{v}}}{c^2}$$

we obtain

$$\dot{\gamma} = -\frac{e\mathbf{E}\cdot\mathbf{v}}{mc^2}. \tag{3}$$

For the given fields equation (3) becomes

$$\dot{\gamma} = -\frac{eE_0}{mc^2}\{(1-ay)\dot{y}+az\dot{z}\}. \tag{4}$$

From equation (1)

$$\frac{d}{dt}(\gamma\dot{z}) = -\frac{e}{m}E_0az. \tag{5}$$

Clearly, if γ is treated as a constant, we have simple harmonic motion of period $2\pi(\gamma m/eE_0a)^{1/2}$. In practice, we shall be interested only in situations where many revolutions of the xy motion occur in one period of the z motion. The value of γ which is relevant to equation (5) is thus its time average $\bar{\gamma}$ taken over a period of the xy motion and, to a good approximation, we may take $\bar{\gamma} = \gamma_0$, where $\gamma_0 = (1-v_0{}^2/c^2)^{-1/2}$, v_0 being the magnitude of the velocity of emission of the electron from the source. This approximation will be discussed in detail in the appendix.

The period of the z motion then is

$$\tau_z \simeq 2\pi \left(\frac{\gamma_0 m}{eE_0 a}\right)^{1/2}.$$

For weak electric fields such as occur in the present experiment, the period τ of the xy motion differs only slightly from the cyclotron period $2\pi/\omega_0 = 2\pi\gamma_0 m/eB_0$ (see below), where $\omega_0 = eB_0/\gamma_0 m$ is the cyclotron frequency. Thus, to a sufficiently good approximation, the number of periods of the xy motion which occur during one period of the z motion is

$$\frac{\tau_z}{\tau} \simeq B_0 \left(\frac{e}{\gamma_0 m E_0 a}\right)^{1/2}. \tag{6}$$

We may use this equation to estimate a suitable value for a. If an electron for which $\dot{z}_0 \neq 0$ is to be brought back to the xy plane after, say, 100 periods of the xy motion, then $\tau_z/\tau = 200$. In the present experiment it is required that, when $E_0 = 0$, electrons of energy about 500 kev form circular orbits of radius about 4 cm. Therefore $\gamma_0 \simeq 2$ and $B_0 \simeq 10^{-1}$ wb m^{-2}. Typically, $E_0 \simeq 5 \times 10^3$ v m^{-1}. Substitution of these values into equation (6) gives

$$a \simeq 5\ \mathrm{m}^{-1} \tag{7}$$

corresponding to a maximum value of about 20 cm for y.

In practice, we shall be interested in particles which do not depart from the xy plane by more than about ± 1 cm. In the appendix it is shown that, for such particles, $\dot{z} \lesssim 10^{-3} v_0$. Hence, for the present experiment, the term $az\dot{z}$ in equation (4) is negligible. Henceforth we disregard entirely the z motion.

Integration of equation (4) then gives

$$\gamma = \gamma_0\{1 - k(1 - \tfrac{1}{2}ay)y\} \tag{8}$$

where $k = eE_0/\gamma_0 mc^2$.

Integrating the x component of equation (1), we have

$$\dot{x} = \frac{v_0 \sin\phi - \omega_0 y}{1 - k(1 - \frac{1}{2}ay)y}. \tag{9}$$

The y component of velocity may then be found from the relation $\dot{x}^2 + \dot{y}^2 = v^2$. Thus, from equations (8) and (9),

$$\dot{y}^2 = [v_0^2 \cos^2\phi - 2(c^2k - v_0\omega_0 \sin\phi)y - \{\omega_0^2 - c^2k(a+k)\}y^2 - c^2k^2ay^3 + \tfrac{1}{4}c^2k^2a^2y^4]\{1 - k(1 - \tfrac{1}{2}ay)y\}^{-2}. \tag{10}$$

Substituting the same values of E_0 and γ_0 as before, we find

$$k \simeq 5 \times 10^{-3}\ \mathrm{m}^{-1}. \tag{11}$$

The magnitude of y will not exceed about 5 cm in the present experiment. Using this and the values of a and k given in equations (7) and (11),

$$ky \simeq 2{\cdot}5 \times 10^{-4} \ll 1$$
$$kay^2 \simeq 6 \times 10^{-5} \ll 1$$

and so, in equation (10), the terms in y^3 and y^4 may be neglected. However, we retain the term $c^2k^2y^2$, so that expressions derived below may have the correct form in the limit $a \to 0$.

We define the *step size* S to be the change in the x coordinate of the particle between, for example, two successive crossings of the x axis in the same sense. Thus

$$S = \oint dx = \oint \left(\frac{\dot{x}}{\dot{y}}\right) dy.$$

Hence, from equation (9) and from equation (10), omitting the terms in y^3 and y^4, we find†

$$S = \frac{2\pi c^2 k\{\omega_0 - (a+k)v_0 \sin\phi\}}{\{\omega_0^2 - c^2 k(a+k)\}^{3/2}}. \tag{12}$$

With $a = 0$ and k small (homogeneous, weak electric field), S reduces to the value $2\pi E_0/\omega_0 B_0$. When $\omega_0^2 = c^2k^2$ (i.e. when $E_0/cB_0 = 1$), the step size becomes infinite, indicating that the trajectory is no longer closed. In practice, we shall be interested only in situations where $\omega_0^2 \gg c^2k^2$, i.e. where $(E_0/cB_0)^2 \ll 1$.

The *period* of the motion is defined to be

$$\tau = \oint \frac{\{(dx)^2 + (dy)^2\}^{1/2}}{v} = \oint \frac{dy}{\dot{y}}.$$

Hence, again omitting the terms in y^3 and y^4 in equation (10), we find

$$\tau = \frac{2\pi}{\{\omega_0^2 - c^2k(a+k)\}^{3/2}} \left[\omega_0^2 - kv_0\omega_0 \sin\phi - ka\left\{c^2 - \tfrac{1}{4}v_0^2\cos^2\phi - \frac{3}{4}\frac{(c^2k - v_0\omega_0\sin\phi)^2}{\omega_0^2 - c^2k(a+k)}\right\}\right]. \tag{13}$$

When $a = 0$ and k is small, τ tends to the cyclotron period $2\pi/\omega_0$. As already noted, the step size under these conditions is approximately $2\pi E_0/\omega_0 B_0$; the trajectory therefore approximates closely to a circle drifting with velocity E_0/B_0 in the direction of the positive x axis. As k increases, the trajectory becomes more trochoidal and less like a slowly drifting circle.

The effects of the electrostatic field on the orbital and spin precession motions are discussed in greater detail in § 5.

The *drift velocity* of the electron is defined to be

$$V = S/\tau.$$

From equations (12) and (13),

$$V = c^2k\{\omega_0^2 - (a+k)v_0\sin\phi\}\left[\omega_0^2 - kv_0\omega_0\sin\phi - ka\left\{c^2 - \tfrac{1}{4}v_0^2\cos^2\phi - \frac{3}{4}\frac{(c^2k - v_0\omega_0\sin\phi)^2}{\omega_0^2 - c^2k(a+k)}\right\}\right]^{-1}. \tag{14}$$

† The solution of the differential equation for the trajectory in the present case may be written parametrically as

$$x = x_0 + \frac{Q\theta - (Q^2 + 4PR)^{1/2}\cos\theta}{2\omega_0 R^{3/2}}$$

$$y = \left\{v_0 \sin\phi - \frac{Q + (Q^2 + 4PR)^{1/2}\sin\theta}{2R}\right\}\frac{1}{\omega_0}$$

where x_0 is an arbitrary constant, and

$$P = v_0^2 - \frac{2c^2kv_0\sin\phi}{\omega_0} + \frac{c^2k(a+k)v_0^2\sin^2\phi}{\omega_0^2}$$

$$Q = 2c^2k\left\{1 - \frac{(a+k)v_0\sin\phi}{\omega_0}\right\}\frac{1}{\omega_0}$$

$$R = 1 - \frac{c^2k(a+k)}{\omega_0^2}.$$

The step size is 2π times the coefficient of θ in the expression for x.

It should be noted that the extension of the orbits in the y direction may be obtained by equating to zero the numerator of equation (10).

Using the same values of a and k as previously, and noting that $\omega_0 \simeq 10^{10}$ rad s^{-1}, we find that, to a good approximation, equation (14) reduces to

$$V \simeq \frac{c^2 k}{\omega_0}\left(1 - a\frac{v_0}{\omega_0}\sin\phi\right)$$

or

$$V \simeq \frac{E_0}{B_0}\left(1 - a\frac{mc}{eB_0}(\gamma_0{}^2 - 1)^{1/2}\sin\phi\right). \tag{15}$$

It should be noted that, when $a = 0$ (homogeneous electric field), the drift velocity has the value E_0/B_0, as is well known, and is independent of the angle of emission ϕ and the energy.

3. Spherical aberration

If we write $E_0/B_0 = V_0$ (the drift velocity for particles emitted in the direction $\phi = 0$) then, from equation (15),

$$V_0 - V = \frac{E_0}{B_0}a\frac{mc}{eB_0}(\gamma_0{}^2 - 1)^{1/2}\sin\phi$$

$$= \frac{E_0}{B_0}\delta\sin\phi$$

say, where $\delta = a(mc/eB_0)(\gamma_0{}^2 - 1)^{1/2}$.

Let the total distance between source and analyser be D. The difference in time taken to drift this distance by particles emitted at angles zero and ϕ is thus

$$\Delta t = \frac{D}{V_0}\frac{V_0 - V}{V} = \frac{D}{V_0}\frac{\delta\sin\phi}{1 - \delta\sin\phi}. \tag{16}$$

The rate of change of helicity of a beam of polarized electrons in a uniform magnetic field B_0 is (Bargmann *et al.* 1959)

$$\omega_D = \omega_L - \omega_C = \gamma\omega_C(\tfrac{1}{2}g - 1) = \frac{e}{m}B_0(\tfrac{1}{2}g - 1)$$

where ω_L is the Larmor spin precession frequency, $\omega_C = eB_0/\gamma m$ is the frequency of the orbital motion and g is the electron g factor. The time taken for, say, a beam which is initially longitudinally polarized to become transversely polarized is therefore

$$\tfrac{1}{4}\tau_D = \frac{1}{4}\frac{2\pi}{\omega_D} = \frac{\pi m}{eB_0(g-2)}.$$

When $\Delta t = \frac{1}{2}\tau_D$, particles emitted from the source at $\pm\phi$ will have, at the analyser, spins which are antiparallel to those of particles emitted at $\phi = 0$, and serious depolarization of the analysed beam will result. Hence $\Delta t \simeq \frac{1}{4}\tau_D$ gives an upper limit on Δt. Therefore

$$\frac{D}{V_0}\frac{\delta\sin\phi_{max}}{1 - \delta\sin\phi_{max}} \simeq \frac{\pi m}{eB_0(g-2)}.$$

In the present experiment $\delta \simeq 0{\cdot}15$. Taking $\phi_{max} \simeq \pm 6°$ ($\sin\phi_{max} = \pm 0{\cdot}1$), we have, to a good approximation,

$$D \simeq \frac{\pi V_0}{ac(g-2)(\gamma_0{}^2 - 1)^{1/2}\sin\phi_{max}}.$$

Since the g-factor anomaly $\frac{1}{2}g - 1 \simeq 10^{-3}$, and taking the same values for a, E_0, B_0 and γ_0 as before, we find

$$D \simeq 30\ \text{cm}.$$

Also, since $(D/V_0)/\tau_D \simeq 18$, we conclude that, with the arrangement described, it should be possible to observe at least the first eighteen periods of helicity precession of electrons in a uniform magnetic field.

4. Chromatic aberration

Because of the $(\gamma_0{}^2-1)^{1/2}$ term in δ, the drift velocity for a given angle of emission varies with particle energy whenever $\phi \neq 0$. To a good approximation, the fractional change in drift velocity produced by a variation in particle energy is, in magnitude,

$$\frac{\Delta V}{V} = a\frac{mc}{eB_0}\sin\phi\,\Delta\{(\gamma_0{}^2-1)^{1/2}\}$$

so that the time difference in drifting a distance D is

$$\Delta t' = \frac{D}{V}\frac{\Delta V}{V} \simeq \frac{D}{V_0}\frac{\Delta V}{V}.$$

The corresponding time difference Δt for the spherical aberration is given by equation (16). Hence

$$\frac{\Delta t'}{\Delta t} \simeq \frac{\Delta\{(\gamma_0{}^2-1)^{1/2}\}}{(\gamma_0{}^2-1)^{1/2}} = \frac{\gamma_0\Delta\gamma_0}{\gamma_0{}^2-1}.$$

For a change in particle energy of ± 100 kev from 500 kev, $\Delta\gamma_0 \simeq 0{\cdot}2$, and hence $\Delta t'/\Delta t \simeq 0{\cdot}13$. From this we conclude that, for the above-considered energy range, depolarization of the beam at the analyser due to chromatic aberration is insignificant compared with that caused by spherical aberration.

5. Effects of the electrostatic field on the orbital and spin precession frequencies

The effects of unwanted electrostatic field gradients on the precision of previous *g*-factor measurements have been discussed by Wilkinson and Crane (1963), Schupp *et al.* (1961), Nelson *et al.* (1959), and Liebes and Franken (1959). In general, both the orbital and the spin precession frequencies are affected.

For the present experiment equation (13) shows that the angular frequency of the orbital motion, $2\pi/\tau$, deviates from the value ω_0 by an amount of order

$$\frac{\frac{1}{2}c^2ka}{\omega_0} = \frac{\frac{1}{2}aE_0}{B_0} \simeq 10^5\ \text{rad s}^{-1}$$

whereas $\omega_0 \simeq 10^{10}$ rad s^{-1}.

According to G. W. Ford (unpublished, quoted by Nelson *et al.* 1959), the spin precession frequency in electric and magnetic fields **E** and **B** of an electron with an anomalous magnetic moment is

$$\boldsymbol{\omega}_L = -\frac{e}{\gamma m}\left(\mathbf{B}-\frac{\gamma}{\gamma+1}\frac{\mathbf{v}\wedge\mathbf{E}}{c^2}\right) - \frac{e(\frac{1}{2}g-1)}{m}\left(\mathbf{B}-\frac{\gamma-1}{\gamma}\frac{\mathbf{B}\,.\,\mathbf{v}}{v^2}-\frac{\mathbf{v}\wedge\mathbf{E}}{c^2}\right)$$

where e, m, γ and c have the usual meanings.

In the present experiment $\mathbf{B}.\mathbf{v} \simeq 0$, and the appropriate value of $v_x\mathbf{E}$ is its average over one orbital period, i.e. approximately $E_0{}^2/B_0$. Hence, since the anomaly $\frac{1}{2}g-1 \simeq 10^{-3}$, the fractional change in the spin precession frequency due to the electrostatic field is of order $(E_0/cB_0)^2 \simeq 2{\cdot}5\times10^{-8}$, which is entirely negligible.

As discussed in § 3 above, we require to measure the difference frequency $\omega_D = \gamma_0\omega_0(\frac{1}{2}g-1)$ between the angular frequencies of the orbital and spin precession motions. Thus, in the present experiment, $\omega_D \simeq 2\times10^7$ rad s^{-1}. Because of their effect on the orbital frequency, we note that electrostatic field gradients may produce a shift in

ω_D of about 1 part in 200. However, since we aim to *compare* the difference frequencies for electrons and positrons rather than to measure ω_D precisely for each particle, the shift is unimportant in the present experiment.

Equally, it should be noted that the actual value of the electrostatic field inhomogeneity parameter a obtaining in the apparatus need not be determined.

6. Realization of the required electrostatic field

The equipotentials $\Phi(y, z)$ corresponding to the field $\mathbf{E} = (0, E_0(1-ay), E_0az)$ are given by

$$\Phi = -E_0\{y-\tfrac{1}{2}a(y^2-z^2)\}+\text{const}.$$

Initially, we wish to determine where the equipotentials cross the y axis, so we put $z = 0$.

Let $\Phi = 0$ at $y = 0$, and let $\Phi = -\Phi_0$ at $y = y_0$, where Φ_0 and y_0 are positive constants. Then

$$\frac{\Phi}{\Phi_0} = -\frac{y(1-\frac{1}{2}ay)}{y_0(1-\frac{1}{2}ay_0)}.$$

Hence, as $ay_0 \to 2$, $\Phi \to \infty$. In general, the potential has a turning value at $1/a$. So that the electric field should not change sign within the region of interest, we have the restriction

$$ay_0 < 1.$$

For the present experiment $a = 5\times10^{-2}\ \text{cm}^{-1}$. If we take $y_0 = 10$ cm, the equation of the equipotentials becomes

$$y^2-40y-z^2-\frac{300\Phi}{\Phi_0} = 0$$

where y and z are in centimetres.

The corresponding equipotentials are shown as broken lines in figure 2. In practice, appropriately shaped metal plates are inserted in positions which correspond to $\Phi = \pm\Phi_0$, and these electrodes are connected to a suitable, symmetrical voltage supply. To minimize the effects of fringing fields, additional metal plates, each at a potential appropriate to its position, are placed above and below the mid-plane of the system. The source, situated at the origin, is at earth potential.

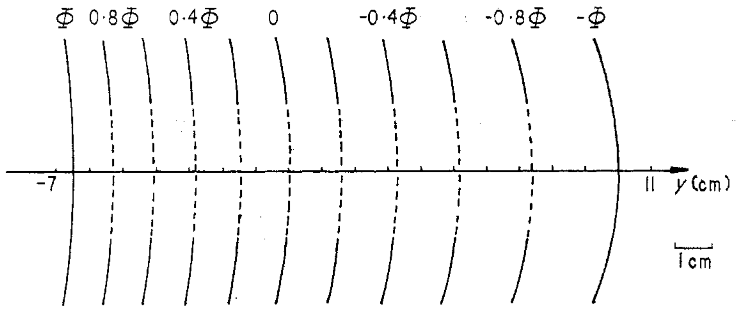

Figure 2. Electrode system to achieve the field $\mathbf{E} = (0, E_0(1-ay), E_0az)$. Broken lines indicate the equipotentials.

An investigation using resistance paper and a commercial field plotter suggests that, with nine correcting electrodes (as shown in figure 2), departure of the equipotentials from their theoretical positions is negligibly small.

7. Conclusions

It is well known that charged particles which are subject to static magnetic and electric fields $(0, 0, B_0)$ and $(0, E_0, 0)$ (where B_0 and E_0 are positive constants and $(E_0/cB_0)^2 \ll 1$,

c being the velocity of light) move, in the xy plane, in orbits which approximate closely to circles drifting in the positive x direction. The drift velocity is E_0/B_0, regardless of the direction of injection of the particles into the system and of the energy of the particles. Since there is no z component of force on the particles, any which possess a component of velocity in the z direction will spiral out of the xy plane and be lost.

We have analysed the effects of modifying the electric field, so that it becomes $(0, E_0(1-ay), E_0az)$, where a is a positive constant. It has been shown that the desired focusing into the xy plane is obtained at the expense of causing the drift velocity to be a function both of the direction of injection of the particle into the system (the source of particles being at the origin) and of the energy of the particle.

The applicability of such a distribution of fields to an electron–positron g-factor comparison experiment has been discussed in detail. For 500 kev particles emitted into a magnetic field of about 0·1 wb m^{-2} with velocity v_0 at an angle ϕ to the negative y direction, we have shown that the drift velocity is approximately

$$\frac{E_0}{B_0}\left\{1-a\frac{mc}{eB_0}(\gamma_0{}^2-1)^{1/2}\sin\phi\right\}$$

where $-e$ and m are the charge and mass of the particle and $\gamma_0 = (1-v_0{}^2/c^2)^{-1/2}$. It is concluded that, in spite of the undesirable dependence of the drift velocity on ϕ and γ_0, such a comparison experiment is a practical possibility.

Appendix

The z component of the equation of motion is (equation (5))

$$\frac{d}{dt}(\gamma\dot{z}) = -\frac{e}{m}E_0az$$

i.e.

$$\ddot{z}+\frac{\dot{\gamma}}{\gamma}\dot{z}+\frac{eE_0a}{\gamma m}z = 0. \tag{A1}$$

We assume $\tau_z \gg \tau$ and replace γ by $\bar{\gamma}$, where

$$\bar{\gamma} = \frac{\gamma_0}{\tau}\oint\frac{1-ky+\frac{1}{2}aky^2}{\dot{y}}dy$$
$$\simeq \gamma_0\left[1+k\{k(c^2+\tfrac{1}{2}v_0{}^2)+\tfrac{1}{4}av_0{}^2\}\frac{1}{\omega_0{}^2}-\frac{kv_0\sin\phi}{\omega_0}+\frac{\frac{1}{2}kav_0{}^2\sin^2\phi}{\omega_0{}^2}\right].$$

In the present experiment, taking $k = 1$ m^{-1} (as an upper limit) and $a = 5$ m^{-1}, we find $c^2k^2/\omega_0{}^2 \simeq 10^{-3}$ and $kv_0/\omega_0 \simeq 3\times10^{-2}$. Thus, if $\sin\phi \simeq 0{\cdot}1$,

$$\bar{\gamma} \simeq \gamma_0$$

is a good approximation.

From equation (4),

$$\dot{\gamma} = -\frac{eE_0}{mc^2}\{(1-ay)\dot{y}+az\dot{z}\} = \dot{\gamma}_y+\dot{\gamma}_z$$

where $\dot{\gamma}_y = -eE_0(1-ay)\dot{y}/mc^2$ and $\dot{\gamma}_z = -eE_0az\dot{z}/mc^2$. Replacing $\dot{\gamma}_y$ by its average over one orbital period, we find

$$\bar{\dot{\gamma}}_y = 0.$$

Hence we may replace $\dot{\gamma}/\gamma$ by $\dot{\gamma}_z/\gamma_0$, so that equation (A1) becomes

$$\ddot{z}+\frac{eE_0a}{m\gamma_0}\left(1-\frac{\dot{z}^2}{c^2}\right)z = 0.$$

If $\dot{z}^2/c^2 \ll 1$, the z motion is, to a very good approximation, simple harmonic, of period

$$\tau_z = 2\pi \left\{\frac{m\gamma_0}{eE_0 a}\right\}^{1/2}.$$

Now

$$\dot{z} \leqslant v_0 \sin\psi$$

where ψ is the angle between the initial direction of the particle and the xy plane, given by

$$\tan\psi \simeq \frac{z_0 \tau}{\rho \tau_z}$$

where z_0 is the amplitude of the z motion and ρ is the cyclotron radius. For $\tau_z/\tau \simeq 200$, $\rho \simeq 4$ cm and $z_0 \simeq 1$ cm, we find $\sin\psi \simeq 10^{-3}$, and so the perturbation due to $\dot{\gamma}$ has negligible effect on the z motion.

References

Bargmann, V., Michel, L., and Telegdi, V. L., 1959, *Phys. Rev. Lett.*, **2**, 435–6.
Farago, P. S., Gardiner, R. B., Muir, J., and Rae, A. G. A., 1963, *Proc. Phys. Soc.*, **82**, 493–500.
Liebes, S., and Franken, P., 1959, *Phys. Rev.*, **116**, 633–50.
Nelson, D. F., Schupp, A. A., Pidd, R. W., and Crane, H. R., 1959, *Phys. Rev. Lett.*, **2**, 492–5.
Rich, A., and Crane, H. R., 1966, *Phys. Rev. Lett.*, **17**, 271–5.
Schupp, A. A., Pidd, R. W., and Crane, H. R., 1961, *Phys. Rev.*, **121**, 1–17.
Wilkinson, D. T., and Crane, H. R., 1963, *Phys. Rev.*, **130**, 852–63.